

On partial analyticity of CR mappings

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Abstract. We study the problem of holomorphic extension of a smooth CR mapping from a real analytic hypersurface to a real algebraic set in complex spaces of different dimensions.

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1 Introduction

Let X and Y be real analytic Cauchy - Riemann manifolds in complex affine spaces (of different dimensions, in general), and $f : X \longrightarrow Y$ be a smooth \mathcal{CR} mapping. It is natural to ask under what conditions f is real analytic (and, therefore, extends holomorphically to a neighborhood of X)? The intrinsic tool to study this problem is the reflection principle. It has two major variations: the analytic one (which makes use of the tangent Cauchy - Riemann operators) and the geometric one (involving a study of analytic geometry of the Segre families). In the present paper we assume that Y is a real *algebraic* variety and use the analytic approach. Algebraicity of Y allows to involve additionally methods of commutative algebra. A local character of the problem makes it more convenient to formulate some of our results in terms of germs of \mathcal{CR} mappings.

Let X be a real analytic hypersurface in \mathbb{C}^n , minimal at a point $p \in X$. As usual, the minimality means that X contains no germs of complex hypersurfaces in a neighborhood of p . Let also Y be a real algebraic subset of \mathbb{C}^N i.e. the zero set of finite number of real valued polynomials in \mathbb{C}^N . Fix a point $p \in X$ and consider a germ of a smooth (everywhere below this means C^∞) \mathcal{CR} mapping ${}_p\mathbf{f} : X \longrightarrow Y$ of X to Y . This means that there exists a neighborhood U of p in \mathbb{C}^n and a representative mapping f of ${}_p\mathbf{f}$ defined on $X \cap U$ such that $f(X \cap U) \subset Y$.

Consider also the field $\mathcal{M}_p(X)$ of restrictions to X of germs of meromorphic functions at p and the finite type extension $\mathcal{M}_p(X)({}_p\mathbf{f}_1, \dots, {}_p\mathbf{f}_N)$ of this field generated by the components of the germ ${}_p\mathbf{f}$. Denote by $tr.deg.({}_p\mathbf{f})$ the transcendence degree of this extension over $\mathcal{M}_p(X)$ (see section 2 for definitions). The transcendence degree measures "the degree of non - analyticity" of ${}_p\mathbf{f}$. If $tr.deg.{}_p\mathbf{f}$ is equal to m , we will show that the graph of ${}_p\mathbf{f}$ is contained in a complex $(n + m)$ - dimensional variety in \mathbb{C}^{n+N} . In particular, we will show that ${}_p\mathbf{f}$ is real analytic if and only if $tr.deg.{}_p\mathbf{f} = 0$.

Let $f : X \longrightarrow \mathbb{C}^N$ be a smooth \mathcal{CR} mapping of a real analytic minimal hypersurface. There exists an open dense subset of X where the rank of f achieves its maximal value, which we call the generic rank of f . As usual, by $rank_p\mathbf{f}$ (the rank of the germ ${}_p\mathbf{f}$) we mean a generic rank of its representative mapping; this definition does not depend on the choice of a representative. The

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geometric property of real submanifolds in \mathbb{C}^n which we study in this paper can be described as follows.

Definition 1.1 *Let Y be a real algebraic subset of \mathbb{C}^N and p be a point in Y . Let also r and m be positive integers. We say that Y is (r, m) - flat at p if there exists a real analytic submanifold $M \subset Y$ through p of real dimension r which is biholomorphic to the cartesian product $\Gamma \times D$, where Γ is a real analytic submanifold in a complex space of smaller dimension and D is a domain in \mathbb{C}^m .*

Of course, this definition still makes sense, if Y is just real analytic. In this paper we will use it in the algebraic case only.

One can consider the manifold M in Definition 1.1 (up to biholomorphic equivalence) as a trivial fiber bundle with *holomorphic* fibers and the real analytic manifold Γ as the base. If Y admits such a bundle, holomorphically embedded as a submanifold, then in general one can construct a smooth \mathcal{CR} mapping from X to Y which is not real analytic. For instance, if a manifold $\Gamma \times D$, where Γ is a real submanifold in \mathbb{C}^k and D is a domain in \mathbb{C} , is contained in Y and X is a real sphere, one can consider a \mathcal{CR} mapping from X to $\Gamma \times D$ of the form (g, h) , where $g : X \rightarrow \Gamma$ is a constant mapping and h is smooth, but not real analytic \mathcal{CR} function on X .

The main goal of the paper is to prove the following

Main Theorem *Let $X \subset \mathbb{C}^n$ ($n > 1$) be a real analytic hypersurface, minimal at a point $p \in X$ and $Y \subset \mathbb{C}^N$ be a real algebraic set. Let ${}_p\mathbf{f} : X \rightarrow Y$ be a germ of a smooth \mathcal{CR} mapping at p with $\text{rank}_p \mathbf{f} = r$, $\text{tr.deg.}_p \mathbf{f} = m$. Then in any neighborhood of ${}_p\mathbf{f}(p)$ there are points where Y is (r, m) - flat.*

This result means that the existence of bundles described by Definition 1.1 is the only reason for the non - analyticity of a \mathcal{CR} mapping and, moreover, the "degree of non - analyticity" does not exceed the complex dimension of fibers. In order to use this theorem for the study of analytic properties of \mathcal{CR} mappings it suffices to find the obstacles for a holomorphic embedding in Y of such bundles. This gives an upper estimate for the transcendence degree $\text{tr.deg.}({}_p\mathbf{f})$. The following result is a corollary of Main Theorem (see section 4).

Theorem 1.2 *Let $X \subset \mathbb{C}^n$ ($n > 1$) be a real analytic hypersurface, $Y \subset \mathbb{C}^N$ be a real algebraic set and ${}_p\mathbf{f} : X \rightarrow Y$ be a germ of a smooth \mathcal{CR} mapping at $p \in X$. Suppose that X is minimal at p and denote by d the maximal dimension of complex analytic varieties contained in Y in a neighborhood of ${}_p\mathbf{f}(p)$. Then $\text{tr.deg.}({}_p\mathbf{f}) \leq d$ and the graph of ${}_p\mathbf{f}$ is contained in a complex $(n + d)$ - dimensional analytic variety. In particular, if Y does not contain complex analytic varieties of positive dimension in a neighborhood of ${}_p\mathbf{f}(p)$, then ${}_p\mathbf{f}$ is real analytic.*

The estimate of the transcendence degree obtained in Theorem 1.2 may be considered as a result on partial analyticity of ${}_p\mathbf{f}$. If, additionally, X is algebraic, a result similar to Theorem 1.2 was obtained in [4]; however, there is an important difference. In [4] the transcendence degree is considered over the field of *rational* functions in \mathbb{C}^n . Hence, the estimate on the transcendence degree concerns *algebraic* properties of f . The proof of our results is more delicate since we assume X to be only real analytic and purely algebraic methods are not enough. Analytic part of this paper is based on the ideas of [13].

Even in the case when Y does not contain complex analytic varieties of positive dimension, Theorem 1.2 is new. We mention here some special cases of Theorem 1.2 which were previously considered by other authors:

- 1) X is a real analytic strictly pseudoconvex hypersurface in \mathbb{C}^n and Y is a real sphere in \mathbb{C}^N ([13]).
- 2) X is a real analytic strictly pseudoconvex hypersurface in \mathbb{C}^n and Y is a real algebraic set in \mathbb{C}^N with $d = 0$ (A.Pushnikov [15]).
- 3) X and Y are real algebraic strictly pseudoconvex hypersurfaces (X.Huang [5]).
- 4) X is a real algebraic hypersurface in \mathbb{C}^n containing no complex analytic varieties of positive dimension and Y is the sphere in \mathbb{C}^{n+1} (M.S.Baouendi, X.Huang and L.P.Rothschild [2]).

We point out that the result of Theorem 1.2 is new even in the equidimensional case. In particular, we have the following

Corollary 1.3 *Let X and Y be real hypersurfaces in \mathbb{C}^n . Assume that X is real analytic and minimal and Y is real algebraic. Let $f : X \rightarrow Y$ be a C^∞ smooth \mathcal{CR} mapping. If Y contains no complex analytic varieties of positive dimension, then f is real analytic.*

In the case when X is real algebraic and holomorphically non - degenerate, a similar result was obtained by M.Baouendi, X.Huang and L.Rothschild [2].

Other applications of Main Theorem concern germs of \mathcal{CR} mappings of maximal rank. This assumption allows to weaken the restrictions on Y .

Recall that a real hypersurface M is called holomorphically degenerate at a point $a \in M$ if there exists a non-zero germ of a holomorphic vector field tangent to M at a ; M is called holomorphically non - degenerate in a neighborhood U of $p \in M$, if it is not holomorphically degenerate at every point of $U \cap M$. Clearly, $M \cap U$ is holomorphically non - degenerate, if and only if in any neighborhood of every point $a \in M \cap U$ it is not locally biholomorphic to the cartesian product $\Gamma \times D$ where Γ is a real analytic hypersurface a complex space of smaller dimension and D is a domain in \mathbb{C} . One can naturally extend this definition as follows. Let M be a real analytic manifold in a neighborhood of a point $p \in \mathbb{C}^n$. We say that M is d -holomorphically non - degenerate in a neighborhood U of p , if in any neighborhood of any point of $M \cap U$ it is not biholomorphic to a cartesian product $\Gamma \times D$ where D is a domain in \mathbb{C}^d . Notice that a d - holomorphically non - degenerate manifold can contain complex analytic varieties of dimension $\geq d$.

Corollary 1.4 *Let $X \subset \mathbb{C}^n$ be a real analytic hypersurface minimal at a point $p \in X$, and Y be a real algebraic manifold in \mathbb{C}^N , d -holomorphically non - degenerate in a neighborhood of a point $p' \in Y$. Let ${}_p\mathbf{f} : X \rightarrow Y$ be a germ of a smooth \mathcal{CR} submersion with ${}_p\mathbf{f}(p) = p'$. Then $\text{tr.deg.}({}_p\mathbf{f}) < d$ and the graph of ${}_p\mathbf{f}$ is contained in a complex analytic variety of dimension $< n + d$.*

In particular, we have

Corollary 1.5 *Let X and Y be real hypersurfaces in \mathbb{C}^n . Assume that X is real analytic and minimal at $p \in X$ and Y is real algebraic and holomorphically non - degenerate. Let $f : X \rightarrow Y$ be a smooth \mathcal{CR} mapping of generic rank n . Then f is real analytic.*

If both X and Y are algebraic and holomorphically nondegenerate, a similar result was obtained by M.S.Baouendi, X.Huang and L.P.Rothschild [2]. By the example of P.Ebenfelt (see [2]) the minimality condition in Corollary 1.5 cannot be replaced by the holomorphic nondegeneracy. Another special case of Corollary 1.5 (Y is a rigid algebraic hypersurface) was considered by N.Mir [9] and J.Merker and F.Meylan [11]. Related results were also obtained in [8, 10].

2 Outline of the proof of the Main Theorem

Recall some standard constructions from commutative algebra. For any entire ring K denote by $K[x_1, \dots, x_m]$ the ring of polynomials in m variables with coefficients in K . If F is the quotient field of K , we denote by $F(x_1, \dots, x_m)$ the quotient field of $K[x_1, \dots, x_m]$ which is identified with the field of rational fractions over F . In the case where $K = \mathbb{C}$ we identify them with the ring $\mathbb{C}[Z_1, \dots, Z_m]$ (resp. the field $\mathbb{C}(Z_1, \dots, Z_m)$) of polynomial (resp. rational) functions on \mathbb{C}^m . More generally, let $F \hookrightarrow E$ be a field extension and $\alpha_1, \dots, \alpha_n$ be elements of E . We denote by $F(\alpha_1, \dots, \alpha_n)$ the smallest subfield of E , containing the field F and every α_j . Obviously, it coincides with the field of rational fractions in $\alpha_1, \dots, \alpha_n$ with coefficients in F . We say that E is finitely generated over F if there exists a finite family of elements $\alpha_1, \dots, \alpha_n$ in E such that $E = F(\alpha_1, \dots, \alpha_n)$. Recall that a finitely generated extension $E = F(\alpha_1, \dots, \alpha_n)$ is algebraic over F if and only if every α_i is algebraic over F . If S is a subset of E , we still denote by $F(S)$ the smallest subfield of E containing S and say that S generates E .

Let $F \hookrightarrow E$ be a field extension and S be a subset of E . We say that S is algebraically independent over F if for every $s_1, \dots, s_m \in S$ and every polynomial $P \in F[x_1, \dots, x_m]$ the equality $P(s_1, \dots, s_m) = 0$ implies $P = 0$. A finite subset $S \in E$, which is algebraically independent over F and maximal with respect to the inclusion ordering, is called a transcendence base of E over F . In this case E is algebraic over $F(S)$. Any two transcendence bases of E have the same cardinality which is called the degree of transcendence of E over F and is denoted by $tr.deg.(E/F)$. Moreover, if S generates E and S' is a subset of S , algebraically independent over F , then there exists a transcendence base B of E such that $S' \subset B \subset S$. We refer the reader to [7] for proofs of the above statements.

We describe now the general scheme of the proof of the Main Theorem. Let X be a generic real analytic manifold in a neighborhood of a point $p \in X$ in \mathbb{C}^n and Y be a real analytic subset in a neighborhood of a point $p' \in \mathbb{C}^N$. We denote by \mathcal{O}_p the ring of germs of holomorphic functions at p and by $\mathcal{O}_p(X)$ the ring of germs of their restrictions to X (since X is generic, these rings are isomorphic). Let \mathcal{M}_p be the field of germs of meromorphic functions at p (i.e. the quotient field of \mathcal{O}_p) and $\mathcal{M}_p(X)$ the field of germs of their restrictions on X . If D is a domain in \mathbb{C}^n , $\mathcal{O}(D)$ denotes the ring of holomorphic functions on D and $\mathcal{M}(D)$ the field of meromorphic functions on D i.e. the field of sections of the sheaf of germs of meromorphic functions on D . Similarly, $\mathcal{O}(X)$ (resp. $\mathcal{M}(X)$) denote the ring (resp. field) of their restrictions to X . For every function h , meromorphic on Ω , there exists a complex analytic set in Ω such that h is holomorphic outside of this set. The smallest analytic subset of Ω with this property is called the set of singular points of h and is denoted by $Sing_h$. It is well known that every meromorphic function on a Stein domain Ω is as a quotient of two holomorphic functions in Ω .

We denote by ${}_p C_{\mathcal{CR}}^\infty(X)$ the ring of germs of C^∞ smooth \mathcal{CR} functions on X at p and by $({}_p C_{\mathcal{CR}}^\infty(X))^N$ its N -th cartesian power. By a germ of a smooth \mathcal{CR} mapping ${}_p \mathbf{f} : X \rightarrow Y$ we mean an element ${}_p \mathbf{f} \in ({}_p C_{\mathcal{CR}}^\infty(X))^N$ such that for some neighborhood U of p in \mathbb{C}^n there exists a representative mapping f of ${}_p \mathbf{f}$, smooth and \mathcal{CR} on $X \cap U$ and satisfying $f(X \cap U) \subset Y$. As in the previous section, we consider the field extension $\mathcal{M}_p(X)({}_p \mathbf{f}_1, \dots, {}_p \mathbf{f}_N)$ of the field $\mathcal{M}_p(X)$ of germs of meromorphic functions at p finitely generated by components of ${}_p \mathbf{f}$ and denote by $tr.deg.({}_p \mathbf{f})$ its transcendence degree over $\mathcal{M}_p(X)$.

This means that for $m = tr.deg.({}_p \mathbf{f})$ there exist integers $1 \leq i_1 < \dots < i_m \leq N$ such that $({}_p \mathbf{f}_{i_1}, \dots, {}_p \mathbf{f}_{i_m})$ is a transcendence base of $\mathcal{M}_p(X)({}_p \mathbf{f}_1, \dots, {}_p \mathbf{f}_N)$ over $\mathcal{M}_p(X)$. After an eventual

renumeration we assume without loss of generality that $i_1 = 1, \dots, i_m = m$. We will use the notation ${}_p\mathbf{f} = ({}_p\mathbf{g}, {}_p\mathbf{h})$, where ${}_p\mathbf{g} = ({}_p\mathbf{g}_1, \dots, {}_p\mathbf{g}_m) = ({}_p\mathbf{f}_1, \dots, {}_p\mathbf{f}_m)$ and ${}_p\mathbf{h} = ({}_p\mathbf{h}_1, \dots, {}_p\mathbf{h}_{N-m}) = ({}_p\mathbf{f}_{m+1}, \dots, {}_p\mathbf{f}_N)$. Denote the coordinates in \mathbb{C}^N by $Z' = (z', w')$, $z' = (z'_1, \dots, z'_m)$ and $w' = (w'_1, \dots, w'_{N-m})$.

Fix a representative mapping $f = (g, h)$ of ${}_p\mathbf{f}$ in a small enough connected neighborhood U of p in X . The first step of our construction is the embedding of the graph Γ_f to some complex variety, canonically defined by ${}_p\mathbf{f}$.

Since $\{{}_p\mathbf{g}_1, \dots, {}_p\mathbf{g}_m\}$ is an algebraically independent system over the ring $\mathcal{M}_p(X)$, the substitution morphism $\mathcal{O}_p(X)[x_1, \dots, x_m] \rightarrow \mathcal{O}_p(X)[{}_p\mathbf{g}_1, \dots, {}_p\mathbf{g}_m]$ is a ring isomorphism. By the definition of a transcendence basis and the Gauss lemma there are non-zero irreducible polynomials Q_j in the ring $\mathcal{O}_p(X)[{}_p\mathbf{g}_1, \dots, {}_p\mathbf{g}_m][x]$ such that $Q_j({}_p\mathbf{h}_j) = 0$ (see [7], Ch.V, Th.10). Represent every polynomial Q_j in the form $Q_j = \sum_{k=0}^{N_j} {}_p\mathbf{q}_{jk}(Z, {}_p\mathbf{g})x^k$, where $\deg Q_j = N_j > 0$ and ${}_p\mathbf{q}_{jk} \in \mathcal{O}_p(X)[z']$. Denote by q_{jk} corresponding representatives on $U \times W$, where W is a neighborhood of $g(p)$ in \mathbb{C}^m . We associate with them holomorphic functions

$$\hat{Q}_j(Z, Z') := \sum_{k=0}^{N_j} q_{jk}(Z, z')w_j'^k \quad (1)$$

and a complex analytic variety $\mathcal{A}_{p\mathbf{f}}$ in a neighborhood $U \times U'$ of $(p, f(p))$ in $\mathbb{C}^n \times \mathbb{C}^N$, defined as the set of their common zeros:

$$\mathcal{A}_{p\mathbf{f}} = \{(Z, Z') : \hat{Q}_j(Z, Z') = 0, j = 1, \dots, N - m\} \quad (2)$$

where every \hat{Q}_j represents an irreducible polynomial in w_j' over the ring $\mathcal{O}_p(X)[z'_1, \dots, z'_m]$.

The graph Γ_f of f is contained in $\mathcal{A}_{p\mathbf{f}}$. Let

$$\lambda : \mathbb{C}^n(Z) \times \mathbb{C}^m(z') \times \mathbb{C}^{N-m}(w') \rightarrow \mathbb{C}^n(Z) \times \mathbb{C}^m(z')$$

be the natural projection. Consider a complex analytic variety \mathcal{V} in a neighborhood $U \times W$ of the point $(p, g(p))$ in $\mathbb{C}^n \times \mathbb{C}^m$ defined by $\cup_s \{D_s(Z, z') = 0\}$, where $D_s \in \mathcal{O}(U)[z']$ is the discriminant of \hat{Q}_s with respect to w_j' . Then the restriction $\lambda : \mathcal{A}_{p\mathbf{f}} \setminus \lambda^{-1}(\mathcal{V}) \rightarrow \mathbb{C}^n \times \mathbb{C}^m$ is a local biholomorphism. Denote by $\Gamma_{p\mathbf{g}}$ the graph of g in $U \times W$. Since every D_s is an element of $\mathcal{O}(U)[z'_1, \dots, z'_m]$, the algebraic independence of the system $\{{}_p\mathbf{g}_1, \dots, {}_p\mathbf{g}_m\}$ over $\mathcal{O}_p(X)$ implies that $\Gamma_{p\mathbf{g}}$ is not contained in \mathcal{V} . Consider the set

$$\Sigma = \cup_s \{Z \in X \cap U : D_s(Z, g(Z)) = 0\} \quad (3)$$

Then for any $a \in X \setminus \Sigma$ the point $(a, g(a))$ is not in \mathcal{V} . We will show in the next section (Lemma 3.1) that Σ is of measure 0 (with respect to X).

Remark. In general the variety $\mathcal{A}_{p\mathbf{f}}$ is reducible. Let $\tilde{\Gamma}_f = \{(Z, Z') : Z \in X \setminus \Sigma, Z' = f(Z)\}$. Since $\lambda : \mathcal{A}_{p\mathbf{f}} \rightarrow U \times V$ is a local biholomorphism near any $(Z, Z') \in \tilde{\Gamma}_f$, the union of all irreducible components of $\mathcal{A}_{p\mathbf{f}}$ with non - empty intersections with $\tilde{\Gamma}_f$ is a complex analytic set in $U \times U'$ of pure dimension $n + m$, containing the graph Γ_f (one can easily show that, in fact, there is only one such component, but we do not need this fact).

The next step is to study the local geometry of $\mathcal{A}_{p\mathbf{f}}$ in a neighborhood of a point $a \in (X \cap U) \setminus \Sigma$. Let $\pi : \mathbb{C}^n \times \mathbb{C}^N \longrightarrow \mathbb{C}^n$ and $\pi' : \mathbb{C}^n \times \mathbb{C}^N \longrightarrow \mathbb{C}^N$ be the natural projections. We denote by $\mathcal{A}_{p\mathbf{f}}|X$ the "restriction" of $\mathcal{A}_{p\mathbf{f}}$ to X defined by the additional assumption $Z \in X$ in equations (2) i.e. $\mathcal{A}_{p\mathbf{f}} \cap \pi^{-1}(X)$. Obviously, $\pi(\mathcal{A}_{p\mathbf{f}}|X) \subset X$.

It follows from the implicit function theorem that there exist a neighborhood U_a of a in \mathbb{C}^n , a neighborhood V'_a of $g(a)$ in \mathbb{C}^m and a neighborhood V''_a of $h(a)$ in \mathbb{C}^{N-m} such that $\mathcal{A}_{p\mathbf{f}}|X$ can be represented in $U_a \times V_a$ with $V_a = V'_a \times V''_a$ in the form $w' = H(Z, z')$, $Z \in X \cap U_a$, where H is holomorphic in $U_a \times V'_a$. Hence the triple $\{\mathcal{A}_{p\mathbf{f}}|X \cap (U_a \times V_a), \pi, X \cap U_a\}$ has the structure of a trivial fiber bundle over $X \cap U_a$ with holomorphic fibers $\pi^{-1}(Z) = \{Z' = (z', w') : w' = H(Z, z')\}$ of complex dimension m .

The crucial question is whether the projection π' takes the fibers of π to Y ? Thus we need the following condition :

$$\pi'((\mathcal{A}_{p\mathbf{f}}|X) \cap (U_a \times V_a)) \subset Y \quad (4)$$

This condition allows to transfer the structure of the bundle $\{(\mathcal{A}_{p\mathbf{f}}|X) \cap (U_a \times V_a), \pi, X \cap U_a\}$ to the target set Y . Since the fibers of π are complex manifolds and the projection π' is holomorphic in the ambient space, it preserves the complex structure of fibers. Moreover, the restriction of π' to a fiber of π is biholomorphic and this allows to construct many complex subvarieties in Y .

The following proposition shows that the condition (4) implies the (r, m) - flatness of Y and is the key for our approach.

Proposition 2.1 *Suppose that (4) holds. Then there exist a positive integer $r = r(a) \geq \text{rank}_{p\mathbf{f}}$ such that in any neighborhood of $f(a)$ there are points where Y is (r, m) - flat.*

Proof : This is a direct corollary of the rank theorem. Denote by r the maximal rank of the restriction $\pi'|(\mathcal{A}_{p\mathbf{f}}|X) \cap (U_a \times V_a)$ and by S the open dense subset of $(\mathcal{A}_{p\mathbf{f}}|X) \cap (U_a \times V_a)$ where this rank is equal to r . Since the graph of f over X is contained in $\mathcal{A}_{p\mathbf{f}}|X$, r is bigger or equal to the generic rank of f . Fix a point $(Z^0, Z'^0) \in S$, close enough to $(a, f(a))$, and neighborhoods $W_a \subset U_a$ of Z^0 in \mathbb{C}^n and $W'_a \subset V_a$ of Z'^0 in \mathbb{C}^N , such that the intersection $\Omega = (\mathcal{A}_{p\mathbf{f}}|X) \cap (W_a \times W'_a)$ is contained in S . It follows from (4) and the real analytic version of the rank theorem, applied to the restriction $\pi'|\Omega$, that there exist neighborhoods $\hat{W}_a \subset W_a$ of Z^0 in \mathbb{C}^n and $\hat{W}'_a \subset W'_a$ of Z'^0 in \mathbb{C}^N such that $M_a := \pi'(\Omega)$ is an r -dimensional real analytic submanifold in $Y \cap W'_a$. Furthermore, since the restriction of the projection π' on every fiber of π has the maximal rank, there exists a real analytic submanifold $\Gamma_a \subset X$ through Z^0 such that the restriction $\pi' : \Omega \cap \pi^{-1}(\Gamma_a) \longrightarrow M_a$ is a real analytic diffeomorphism. Since the projection π' is holomorphic on the ambient space and $\Omega \cap \pi^{-1}(\Gamma)$ is a trivial fiber bundle over Γ_a with holomorphic fibers, we get desired statement.

Since $(X \cap U) \setminus \Sigma$ is dense in $X \cap U$, this proves the Main Theorem under the additional condition (4).

The goal of the rest of this paper is to show that the condition (4) holds if X is a real analytic minimal hypersurface in \mathbb{C}^n and Y is a real algebraic set in \mathbb{C}^N . We make use of two principal tools: results on meromorphic extension of certain classes of \mathcal{CR} functions and algebraic field extensions. In the next section we discuss the problem of meromorphic extension.

3 Reflection, meromorphic extension and algebraic dependence over intermediate fields

3.1. Intermediate field extensions. Let X be a real analytic generic minimal (in the sense of A.Tumanov [20]) submanifold in a neighborhood of a point $p \in X$ in \mathbb{C}^n . Assume that X is defined by a vector - valued function $\rho = (\rho_1, \dots, \rho_d)$ which is real analytic in a neighborhood of p and $\partial\rho_1 \wedge \dots \wedge \partial\rho_d \neq 0$ at p . Denote by $Z = (z, w)$ with $z = (Z_1, \dots, Z_{n-d})$ and $w = (Z_{n-d+1}, \dots, Z_n)$ the coordinates in \mathbb{C}^n . Assume that $\det(\frac{\partial\rho}{\partial w})(p) \neq 0$. Consider the Cauchy - Riemann operators in a neighborhood of p on X given by

$$\mathcal{L}_k = \frac{\partial}{\partial \bar{z}_k} - \sum_{j=1}^d a_{jk}(Z) \frac{\partial}{\partial \bar{w}_j} \quad (5)$$

with $(a_{jk}) = (\partial\rho/\partial w)^{-1}(\partial\rho/\partial z)$.

Denote by $\mathcal{O}_p^{\mathbb{R}}(X)$ the ring of germs at p of real analytic functions on X and by ${}_p C_{\mathcal{CR}}^{\infty}(X)$ (resp. ${}_p C_{\overline{\mathcal{CR}}}^{\infty}(X)$) the ring of germs at p of C^{∞} smooth \mathcal{CR} (resp. anti \mathcal{CR}) functions on X . If Ω is an open subset of X , we denote by $\mathcal{O}^{\mathbb{R}}(\Omega)$ (resp. $C_{\mathcal{CR}}^{\infty}(\Omega)$) the ring of real analytic (resp. anti \mathcal{CR}) functions on Ω .

We denote by $\mathcal{P}_p(X)$ the set of linear combinations of elements of ${}_p C_{\overline{\mathcal{CR}}}^{\infty}(X)$ with coefficients in $\mathcal{O}_p^{\mathbb{R}}(X)$. It is worthwhile to emphasize that the values of real analytic coefficients and anti \mathcal{CR} functions are always considered at the same point of X , so the elements of $\mathcal{P}_p(X)$ are germs of functions on X at p .

Notice that $\mathcal{P}_p(X)$ is a $\mathcal{O}_p^{\mathbb{R}}(X)$ - module and $\mathcal{L}_j({}_p \mathbf{h}) \in \mathcal{P}_p(X)$ for every germ ${}_p \mathbf{h} \in \mathcal{P}_p(X)$.

We will need the following property of the ring $\mathcal{P}_p(X)$.

Lemma 3.1 *Let X be a generic real analytic manifold in \mathbb{C}^n minimal at a point $p \in X$, and ${}_p \mathbf{h} \in \mathcal{P}_p(X)$. Let U be a connected neighborhood of p in \mathbb{C}^n , h be a representative of ${}_p \mathbf{h}$ on $X \cap U$ and $E = \{Z \in X \cap U : h = 0\}$. Suppose that the intersection of E with every neighborhood of p is of positive measure (with respect to X). Then ${}_p \mathbf{h} = 0$.*

Proof : It follows from [20] that there exist small enough connected neighborhoods $U' \subset U$ of p and an open convex cone \mathcal{V} in \mathbb{R}^d such that each \mathcal{CR} function on $X \cap U$ extends holomorphically to the wedge $\mathcal{W} = \{Z \in U' | r(Z) \in \mathcal{V}\}$.

Consider a foliation of $X \cap U'$ by real analytic maximal totally real manifolds $\{M_t\}_t$ (i.e. $\dim M_t = n$ for any t), where the parameter t is in a domain $D \subset \mathbb{R}^{n-d}$. By the Fubini - Tonelli theorem $F := \{t \in D : \text{mes}(M_t \cap E) > 0\}$ is the subset of D of positive measure. Let $t \in F$ and $h = \sum_{\nu=1}^s a_{\nu} g_{\nu}$ with $a_{\nu} \in \mathcal{O}^{\mathbb{R}}(X \cap U)$, $g_{\nu} \in C_{\overline{\mathcal{CR}}}^{\infty}(X \cap U)$.

Every restriction $a_{\nu}|_{M_t}$ extends antiholomorphically to a neighborhood of M_t and every g_{ν} extends antiholomorphically to \mathcal{W} . Hence, $h|_{M_t}$ coincides with some function \hat{h} which is antiholomorphic \mathcal{W} and smooth up to $M_t \subset X \cap U$. By the boundary uniqueness theorem [3, 14] $\hat{h} \equiv 0$ and thus $h|_{M_t} = 0$. Therefore, $\cup_{t \in F} M_t \subset E$.

Now we can consider another foliation $\{M'_t\}$ of $X \cap U$ such that each $M'_t \cap E$ has positive measure and repeat the previous arguments to conclude that $E = X \cap U$.

In particular, we have the following

Corollary 3.2 *If X is minimal at p , then $\mathcal{P}_p(X)$ is an entire ring.*

This last corollary allows us to introduce the quotient field of the ring $\mathcal{P}_p(X)$ which we denote by $\mathcal{M}_p^*(X)$. Let U be a neighborhood of p in \mathbb{C}^n . A representative h of ${}_p\mathbf{h}$ of $\mathcal{M}_p^*(X)$ in $X \cap U$ is a quotient φ/ψ , where φ and ψ are representatives of germs from $\mathcal{P}_p(X)$ in $X \cap U$. Let $Sing_h := \{Z \in X \cap U : \psi = 0\}$ be the singular set of h . If U is small enough, it is a closed subset of $X \cap U$ of measure 0 and h is a smooth function on $(X \cap U) \setminus Sing_h$. We will call the points of $(X \cap U) \setminus Sing_h$ regular for h . In what follows we say that a subset $E \subset (X \cap U)$ through p is a \mathcal{P} -set, if there exists a non-zero germ ${}_p\mathbf{g} \in \mathcal{P}_p(X)$ which admits a representative g on $X \cap U$ such that $E = \{Z \in X \cap U : g = 0\}$.

3.2. Reflection and meromorphic extension. The following assertion is our main analytic tool.

Proposition 3.3 *Let X be a real analytic hypersurface in \mathbb{C}^n , minimal at $p \in X$. Let ${}_p\mathbf{h} \in \mathcal{M}^*(X)$ and h be its representative in a neighborhood U of p . Assume that there exists a \mathcal{P} -set $E \subset (X \cap U)$ containing $Sing_h$ such that h is a \mathcal{CR} function on $(X \cap U) \setminus E$. Then ${}_p\mathbf{h} \in \mathcal{M}_p(X)$, i.e. ${}_p\mathbf{h}$ is meromorphic.*

Proof : In what follows we denote by $\Delta^k(a, r)$ the polydisc in \mathbb{C}^k centered at a of radius r (we write $\Delta(a, r)$ if $k = 1$ and $\Delta^k(r)$ if $a = 0$). We may assume that $p = 0$, U is a neighborhood of the origin in \mathbb{C}^n of the form $U = \Delta^n(r)$, $r > 0$. In what follows we will replace r by smaller positive numbers when we need; in order to avoid complications of the notations, we keep the same notation r for these numbers. Suppose also that $X \cap U$ is defined by $\{Z \in U : \rho(Z) = 0\}$ where ρ is a real analytic function with $\partial\rho/\partial Z_n \neq 0$ on U and X is minimal at every point of $X \cap U$.

In view of Lemma 3.1 we may assume that $E \cap U$ is of measure 0 (with respect to X). We use the notation $Z = (z, w)$, $z \in \mathbb{C}^{n-1}$, $w \in \mathbb{C}$ for coordinates in \mathbb{C}^n and assume that for every $z \in \Delta^{n-1}(r)$ the linear disc

$$l(z) := \{z\} \times \Delta(r) \tag{6}$$

intersects X transversally at every point.

If $V \subset U$ is a neighborhood in \mathbb{C}^n of a point $a \in (X \cap U)$, V^+ (resp. V^-) denotes the corresponding one-sided neighborhood $\{Z \in V : \rho(Z) > 0\}$ (resp. $\{Z \in V : \rho(Z) < 0\}$).

In view of [19] we may suppose that the following holds:

- (a) there exist a fundamental system of neighborhoods $\{U_s\}_s, \{V_s\}_s$ of the origin with $V_s \subset U_s$ and such that for any s every function holomorphic in U_s^+ extends holomorphically to V_s .
- (b) for every point $a \in X \cap U$ there exist fundamental systems of neighborhoods $\{{}_aV_s\}_s, \{{}_aU_s\}_s$, ${}_aV_s \subset {}_aU_s$ of a such that for any s every function holomorphic in ${}_aU_s^+$ extends holomorphically to ${}_aV_s$ or for any s every function holomorphic in ${}_aU_s^-$ extends holomorphically to ${}_aV_s$.

Let $h = \varphi/\psi$, where the functions φ and ψ are of the form

$$\sum_{\nu=1}^s a_{\nu} g_{\nu} \quad (7)$$

with $a_{\nu} \in \mathcal{O}^{\mathbb{R}}(X \cap U)$, $g_{\nu} \in C_{\mathcal{CR}}^{\infty}(X \cap U)$. We proceed the proof in three steps.

Step 1. Reflection. We begin with the following corollary of the Schwarz reflection principle.

Lemma 3.4 *Let ϕ^{-} be a function, holomorphic on U^{-} and continuous up to $X \cap U$. Then there exists a real analytic function ϕ^{+} on U^{+} continuous up to $X \cap U$ and such that for every $z \in \Delta_{n-1}(r)$ the restriction $\phi^{+}|_{l(z) \cap U^{+}}$ is antiholomorphic and $\phi^{-}|_{(X \cap U)} = \phi^{+}|_{(X \cap U)}$.*

Proof : It follows from the implicit function theorem that there exists a real analytic mapping $\Phi(z, w) : U \rightarrow \Delta$ with the following property: for every $z \in \Delta^{n-1}(r)$ there exists $\delta_z > 0$ such that the mapping $\Phi_z : \Delta(r) \rightarrow \Delta(\delta_z)$, $\Phi_z : w \mapsto \Phi(z, w)$ is a biholomorphism between the domains $\{w \in \Delta(r) : r(z, w) < 0\}$ and $\{\omega \in \Delta(\delta_z) : \text{Im} \omega < 0\}$ and maps the real analytic curve $\{w \in \Delta : r(z, w) = 0\}$ to $[-\delta_z, \delta_z]$. Let $\sigma : \omega \rightarrow \bar{\omega}$ be the standard conjugation in \mathbb{C} . Now it suffices to set $\phi^{+}(z, w) = \phi^{-} \circ \Phi_z^{-1} \circ \sigma \circ \Phi_z(w)$.

Every real analytic coefficient in the representation (7) of φ (resp. ψ) extends in a real analytic way to U^{+} and is holomorphic on every disc (6). By the condition (a) every anti \mathcal{CR} function on $X \cap U^{+}$ extends antiholomorphically to a smaller neighborhood of the origin which we again denote by U . Apply Lemma 3.4 to these antiholomorphic functions in the representation (7) of φ and ψ . We get that they extend to U^{+} as real analytic functions, holomorphic on every disc $l(z) \cap U^{+}$. Finally, we get that φ and ψ extend to real analytic functions φ^{+} and ψ^{+} on U^{+} , which are holomorphic on every disc $l(z) \cap U^{+}$.

Let E' denote the set of points z in $\Delta^{n-1}(r)$ such that the disc $l(z)$ intersects E along a set of positive linear measure. Since every function of the form (7) extends antiholomorphically to $l(z) \cap U^{-}$, it follows from the boundary uniqueness theorem that for every $z \in E'$, one has $(l(z) \cap X) \subset E$. By Fubini - Tonelli theorem, E' is a closed subset of $\Delta^{n-1}(r)$ of measure 0. Hence, $h^{+} := \varphi^{+}/\psi^{+}$ is the quotient of two real analytic in U^{+} functions and for every $z \in \Delta^{n-1}(r) \setminus E'$ the restriction $h^{+}|_{l(z) \cap U^{+}}$ is meromorphic and coincides with h on $l(z) \cap ((X \cap U) \setminus E)$.

Step 2. Meromorphic extension. The next step is to extend h meromorphically to an open dense subset of U^{+} . Let $S = \{Z = (z, w) \in U^{+} : Z \in l(z), z \in E'\}$.

Lemma 3.5 *The function h^{+} is meromorphic on $U^{+} \setminus S$.*

Proof : Let (z_0, w_0) be a point in $U^{+} \setminus S$. Fix a point $a \in l(z_0) \cap ((X \cap U) \setminus E)$. By (b) there exists a one - sided neighborhood V of a such that h extends holomorphically to V . Since h^{+} is meromorphic on every $l(z)$ and coincides with the \mathcal{CR} function h on $(X \cap U) \setminus E$, it is holomorphic on V , if $V \subset U^{+}$, or extends holomorphically to V , if $V \subset U^{-}$. Let us assume that $V \subset U^{+}$ (the other case can be treated similarly).

Fix a point $\omega \in \Delta(r)$ with $(z_0, \omega) \in V$ and $\delta' > 0$ such that $\{z_0\} \times \Delta(\omega, \delta')$ is contained in V . There exists a simply connected domain G in \mathbb{C} such that $\{z_0\} \times G$ is compactly contained

in $U^+ \setminus E$ and contains both $\{z_0\} \times \Delta(\omega, \delta')$ and (z_0, w_0) . Fix $\delta'' > 0$ such that the polydisc $\Delta^{n-1}(z_0, \delta'') \times \Delta(\omega, \delta')$ is contained in V . The function h^+ is holomorphic there and for every fixed $z \in \Delta^{n-1}(z_0, \delta'')$ is meromorphic on $\{z\} \times G$. It follows now from the classical Rothstein lemma [16, 17] that h^+ is meromorphic on $\Delta^{n-1}(z_0, \delta'') \times G$. Thus, h^+ is meromorphic in a neighborhood of any point of $U^+ \setminus S$.

Step 3. Elimination of singularities. We show now that S is a removal singularity for h^+ .

Lemma 3.6 *Let $D \subset \mathbb{C}$ be a domain and φ, ψ be real analytic functions in D . Suppose that $\psi(\zeta^0) \neq 0$ for some $\zeta^0 \in D$ and $h := \varphi/\psi$ is holomorphic in a neighborhood of ζ^0 . Then h is meromorphic in D .*

Proof : It is enough to prove lemma for $D = \Delta = \{|\zeta| < 1\}$ and $\zeta^0 = 0$. Let $\zeta' \in \Delta \setminus \{0\}$ and $\gamma = \{\zeta \in \Delta : \zeta = t\zeta', t \in [0, 1/|\zeta'|]\}$. The restrictions $\varphi|_\gamma$ and $\psi|_\gamma$ admit holomorphic extensions $\hat{\varphi}$ and $\hat{\psi}$ to some neighborhood Ω of γ in Δ . Therefore the function $\hat{\varphi}/\hat{\psi}$ is meromorphic in Ω and coincides with $h = \varphi/\psi$ on γ near 0. By the uniqueness theorem $h = \hat{\varphi}/\hat{\psi}$ in a neighborhood of 0. By analyticity the equality $\varphi\hat{\psi} = \hat{\varphi}\psi$ holds everywhere in Ω and therefore h is meromorphic in $\zeta' \in \Omega$.

Lemma 3.7 *The function h^+ extends meromorphically to U^+ .*

Proof : Let $Z^0 \in S$ and $\tilde{E} = \{Z \in U^+ : \psi^+(Z) = 0\}$. We may assume that $Z^0 = 0$. Moreover, since \tilde{E} is closed and nowhere dense in U^+ , we can choose an affine coordinate system $Z = (Z_1, 'Z)$, $'Z = (Z_2, \dots, Z_n)$ in \mathbb{C}^n with the following properties:

there exist a point $a = (a_1, '0) \in U^+ \setminus \tilde{E}$ and positive $\delta_1 < \delta_2$ such that

- (i) $U_1 := \Delta^n(a, \delta_1) = \Delta(a_1, \delta_1) \times \Delta^{n-1}('0, \delta_1) \subset U^+ \setminus \tilde{E}$,
- (ii) $U_2 := \Delta(a_1, \delta_2) \times \Delta^{n-1}('0, \delta_1) \subset U^+$,
- (iii) $0 \in U_2$.

The function $h^+ = \varphi^+/\psi^+$ is already known to be holomorphic in U_1 . By Lemma 3.6 for any fixed $'Z \in \Delta^{n-1}('0, \delta_1)$ it extends meromorphically to $\Delta(a_1, \delta_2)$ as a function of Z_1 . Hence, by the Rothstein lemma, it is meromorphic in a neighborhood of $Z^0 = 0$.

It follows from (b) that the envelope of holomorphy of U^+ contains a neighborhood of the origin. Therefore, h^+ extends meromorphically to a neighborhood of the origin (see for instance [6, 17, 18]). This completes the proof of Proposition 3.3.

3.3. Algebraic dependence. We will now apply the previous results to analyze the property of algebraic dependence over certain functional fields.

Definition 3.8 *Let $\{p\mathbf{h}_1, \dots, p\mathbf{h}_k\}$ be a finite subset of ${}_pC_{\mathcal{R}}^\infty(X)$. We say that it is algebraically dependent over $\mathcal{M}_p^*(X)$ if there exist a non-zero polynomial ${}_p\mathbf{Q}$ in $\mathcal{M}_p^*(X)[X_1, \dots, X_k]$, a neighborhood U of p , representative functions h_j of ${}_p\mathbf{h}_j$ on $X \cap U$, a representative polynomial Q of ${}_p\mathbf{Q}$ on $X \cap U$ and a \mathcal{P} -set $E \subset X \cap U$ containing the singular set of every coefficient of Q , such that $Q(h_1, \dots, h_k) = 0$ on $(X \cap U) \setminus E$.*

The main result of this section is the following:

Proposition 3.9 *If a finite subset of ${}_pC_{\mathcal{CR}}^\infty(X)$ is algebraically dependent over $\mathcal{M}_p^*(X)$, then it is also algebraically dependent over $\mathcal{M}_p(X)$.*

Proof : Let $\{{}_p\mathbf{h}_1, \dots, {}_p\mathbf{h}_k\} \subset {}_pC_{\mathcal{CR}}^\infty(X)$ be a algebraically dependent system over $\mathcal{M}_p^*(X)$. This means that there exist a neighborhood U of p , a system of representatives $\{h_1, \dots, h_k\}$ on $X \cap U$, a polynomial

$$Q(X_1, \dots, X_k) = \sum_{j=0}^m q_j(Z, \overline{Z}) S_j(X_1, \dots, X_k),$$

where S_j are monomials and every q_j represents a non - zero germ in $\mathcal{M}_p^*(X)$, such that $Q(h_1, \dots, h_k) = 0$ on $(X \cap U) \setminus E$. Here $E \subset X \cap U$ is a \mathcal{P} - set containing the singular set of every q_j . After dividing Q by q_m we additionally have $q_m(Z, \overline{Z}) = 1$ and the degree of the monomial S_m is not zero. We will prove the statement by induction in m .

For $m = 1$ we have $S_1(h_1, \dots, h_k) = 0$, so at least one of ${}_p\mathbf{h}_j$ is zero and the set $\{{}_p\mathbf{h}_1, \dots, {}_p\mathbf{h}_k\}$ is algebraically dependent over $\mathcal{M}_p(X)$.

Now assume that the desired assertion is true for any polynomial containing $\leq m - 1$ terms and apply the Cauchy - Riemann operators (5) to the equation $Q(h_1, \dots, h_k) = 0$. There are two possibilities

- (a) If $\mathcal{L}_s(q_j) = 0$ on $(X \cap U) \setminus E$ for any s and j . Then Proposition 3.3 implies that the coefficients q_j are meromorphic and the set $\{{}_p\mathbf{h}_1, \dots, {}_p\mathbf{h}_k\}$ is algebraically dependent over $\mathcal{M}_p(X)$.
- (b) If there exists j_0 and s_0 such that $\mathcal{L}_{s_0}(q_{j_0}) \neq 0$, we may apply the induction assumption to the polynomial $\sum_{j=0}^{m-1} (\mathcal{L}_{s_0} q_j) S_j$.

We have the following

Corollary 3.10 *Let ${}_p\mathbf{R} \in \mathcal{O}_p^{\mathbb{R}}(X)[w, \overline{w}]$ be a polynomial in $w, \overline{w} \in \mathbb{C}^m$ with real analytic (in $Z \in X$) coefficients and ${}_p\mathbf{g} = ({}_p\mathbf{g}_1, \dots, {}_p\mathbf{g}_m)$ be a germ of a mapping with components in ${}_pC_{\mathcal{CR}}^\infty(X)$ which are algebraically independent over $\mathcal{M}_p(X)$. Suppose that there exist a neighborhood U of p , a representative mapping g of ${}_p\mathbf{g}$ defined on $X \cap U$, a representative polynomial $R \in \mathcal{O}^{\mathbb{R}}(X \cap U)[w, \overline{w}]$ of ${}_p\mathbf{R}$ and a \mathcal{P} - set E in $X \cap U$ such that $R(Z, g, \overline{g})$ vanishes on $(X \cap U) \setminus E$. Then ${}_p\mathbf{R} = 0$ in $\mathcal{O}_p^{\mathbb{R}}(X)[w, \overline{w}]$.*

Proof : We represent R in the form $R = \sum_J b_J(Z, \overline{w}) w^J$ with $b_J \in \mathcal{O}^{\mathbb{R}}(X \cap U)[\overline{w}]$. Assume that there exists a coefficient b_{J_0} such that $b_{J_0}(Z, \overline{g})$ represents a non -zero element in $\mathcal{M}_p^*(X)$. Hence, the system $\{{}_p\mathbf{g}_1, \dots, {}_p\mathbf{g}_m\}$ is algebraically dependent over $\mathcal{M}_p^*(X)$. Then Proposition 3.9 implies that this system is algebraically dependent over $\mathcal{M}_p(X)$: this is a contradiction. Therefore, for any J the coefficient $b_J(Z, \overline{g})$ represents zero in $\mathcal{M}_p^*(X)$. After the complex conjugation we can write $\overline{b_J} = \sum_I c_{JI}(Z) w^I$. If there exist J_0, I_0 such that $c_{J_0 I_0}$ represents a non - zero element in $\mathcal{O}_p^{\mathbb{R}}(X)$, we apply Proposition 3.9 again and obtain a contradiction as above. Thus every $c_{JI} = 0$ in $\mathcal{O}^{\mathbb{R}}(X \cap U)$ and $R(Z, w, \overline{w}) = 0$ in $\mathcal{O}^{\mathbb{R}}(X \cap U)[w, \overline{w}]$. This completes the proof.

4 Completion of the proofs

We suppose that we are in the settings of the Main Theorem and will use the notation of section 2. It suffices to show that (4) holds.

Let the target real algebraic set Y be defined by real polynomials $P'_k(Z', \overline{Z}')$, $k = 1, \dots, q$, $Z' \in \mathbb{C}^N$. Since f takes X to Y , we have $P'_k(f, \overline{f}) = 0, k = 1, \dots, q$.

In this section we use polarization techniques, so we consider real analytic functions as functions in Z, \overline{Z} . Recall also that U is a neighborhood of p in \mathbb{C}^n , chosen in section 2 and Σ is defined by (3).

Proposition 4.1 *There exist polynomials $R_k \in \mathcal{O}^{\mathbb{R}}(X \cap U)[z', \overline{z}']$, $k = 1, \dots, q$ with the following property: for every point $a \in (X \cap U) \setminus \Sigma$ there exist neighborhoods V_1 of $(a, g(a))$ in \mathbb{C}^{n+m} and V_2 of $h(a)$ in \mathbb{C}^{N-m} such that the intersection $(\mathcal{A}_{p, \mathbf{f}} \cap (X \times Y) \cap (V_1 \times V_2))$ is defined by*

$$\{(Z, z', w') \in V_1 \times V_2 : Z \in X, \hat{Q}_j(Z, z', w_j) = 0, R_k(Z, \overline{Z}, z', \overline{z}') = 0, j = 1, \dots, N, k = 1, \dots, q\}$$

Proof : The intersection $\mathcal{A}_{p, \mathbf{f}} \cap (X \times Y)$ is defined by the equations

$$\hat{Q}_j(Z, z', w'_j) = 0, P'_k(Z', \overline{Z}') = 0$$

$Z \in X \cap U$. We can add the conjugate equations $\overline{\hat{Q}_j} = 0$ that does not change the solutions. Consider the complexified system

$$\hat{Q}_j(Z, z', w'_j) = 0, \tilde{Q}_j(\zeta, \tau, \omega_j) = 0, P'_k(z', \tau, w', \omega) = 0 \quad (8)$$

where $j = 1, \dots, N$, $k = 1, \dots, q$, $Z = \overline{\zeta}$, $z' = \overline{\tau}$, $w' = \overline{\omega}$ and $\tilde{Q}_j(\zeta, \tau, \omega_j) := \overline{\hat{Q}_j}(\overline{\zeta}, \overline{\tau}, \overline{\omega}_j)$.

By the elimination theory [21],[12] there exist a neighborhood \mathcal{U} of the point (p, \overline{p}) in $\mathbb{C}^n(Z) \times \mathbb{C}^n(\zeta)$ and functions $R_k(Z, z', \zeta, \tau) \in \mathcal{O}(\mathcal{U})[z', \tau]$, $k = 1, \dots, q$ (which are the resultants of the system (8) with respects to the variables w' and ω) with the following property: for every solution $(Z_0, z'_0, \zeta_0, \tau_0, w_0, \omega_0)$ of (8), such that the leading coefficients of polynomials in (8) are different from zero at (Z_0, z'_0) and (ζ_0, τ_0) , there exists a neighborhood W of $(Z_0, z'_0, \zeta_0, \tau_0, w_0, \omega_0)$ such that for $(Z, z', \zeta, \tau, w, \omega) \in W$ the system (8) is equivalent to

$$\hat{Q}_j(Z, z', w'_j) = 0, \tilde{Q}_j(\zeta, \tau, \omega_j) = 0, R_k(Z, z', \zeta, \tau) = 0$$

In order to construct R_k explicitly, consider the resultant $R_k^1(Z, z', \tau, w'_2, \dots, w'_N, \omega)$ of $\hat{Q}_1(Z, z', w_1)$ and $P'_k(z', \tau, w', \omega)$ with respect to the variable w'_1 . Since $a \in (X \cap U) \setminus \Sigma$, the leading coefficient of $\hat{Q}_1(Z, z', w_1)$ does not vanish at $(a, g(a))$. Hence, the replacing of P'_k by R_k^1 in the system (8) does not change its solutions in a neighborhood of $(a, f(a))$. Consider the resultant $R_k^2(Z, z', \tau, w'_2, \dots, w'_N, \omega)$ of R_k^1 and \hat{Q}_2 with respect to w'_2 , etc. This shows that every P'_k in (8) can be replaced by the function $R_k^N(Z, z', \tau, \omega)$ which is a polynomial in z', τ, ω with coefficients holomorphic in Z in a neighborhood of p . Consider the the resultant $R_k^{N+1}(Z, z', \zeta, \tau, \omega_2, \dots, \omega_N)$ of \hat{Q}_1 and R_k^N with respect to ω_1 , etc. and repeat the same arguments to eliminate variables ω_j . The functions $R_k := R_k^{2N}$ satisfy the desired properties. This completes the proof of proposition.

Now we are able to show (4). It follows from Proposition 4.1 that there exist polynomials $R_k(Z, \overline{Z}, z', \overline{z}') \in \mathcal{O}^{\mathbb{R}}(X \cap U)[z', \overline{z}'], k = 1, \dots, q$ such that for every $a \in (X \cap U) \setminus \Sigma$ there are neighborhoods V_1 of $(a, g(a))$ and V_2 of $h(a)$ such that

$$\mathcal{A}_{p\mathbf{f}} \cap (X \times Y) \cap (V_1 \times V_2) = (\mathcal{A}_{p\mathbf{f}}|_X) \cap \{(Z, z') \in V_1 : R_k(Z, \overline{Z}, z', \overline{z}') = 0, Z \in X, k = 1, \dots, q\}$$

Since the graph Γ_f is contained in $\mathcal{A}_{p\mathbf{f}} \cap (X \times Y)$, we have $R_k(Z, \overline{Z}, g, \overline{g}') = 0$ on $(X \cap U) \setminus \Sigma$. The system $\{\mathbf{g}_1, \dots, \mathbf{g}_m\}$ is algebraically independent over $\mathcal{M}_p(X)$ and $\Sigma \subset (X \cap U)$ is a \mathcal{P} -set, so Corollary 3.10 implies that every R_k represents zero in $\mathcal{O}_p^{\mathbb{R}}(X)[z', \overline{z}']$. Therefore, for any $a \in X \setminus \Sigma$, close enough to p there exists a neighborhood V of the point $(a, f(a))$ in $\mathbb{C}^n \times \mathbb{C}^N$ such that $((\mathcal{A}_{p\mathbf{f}}|_X) \cap V) \cap (X \times Y) = ((\mathcal{A}_{p\mathbf{f}}|_X) \cap V)$. Hence, $\pi'((\mathcal{A}_{p\mathbf{f}}|_X) \cap V)$ is contained in Y , which completes the proof of the Main Theorem.

The estimate of the transcendence degree and the embedding of the graph of the mapping in a complex analytic variety in Theorem 1.2 follow from the Main Theorem and the remark in section 2. In particular, let the dimension of this variety be equal to n i.e. $\text{tr.deg}_p \mathbf{f} = 0$. Since $p\mathbf{f}$ admits a one-side holomorphic extension and is C^∞ , the well known result of Bedford - Bell [1] implies that $p\mathbf{f}$ is real analytic.

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